QM on non-commutative plane

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Abstract

In this paper we describe two simple applications of quantum mechanics on a non-commutative plane. We derive corrections to the standard (commutative) Hamiltonian spectrum for hydrogen-like atom and isotropic linear harmonic oscillator. In the case of LHO we consider the non-commutativity of the momentum operators too.

1 Introduction

In recent years, the method of non-commutative geometry (NC) was developed [1] and applied to various physical situations [2]. By the results of string theory arguments [3] the non-commutative plane have been studied extensively.

Two dimensional non-commutative quantum mechanics (NCQM) is based on a simple modification of commutation relations between the self-adjoint position operators ($\hat{\vec{x}}$) and the self-adjoint momentum operators ($\hat{\vec{p}}$) which satisfy

$$\begin{array}{rcl} [\hat{x}_a, \hat{x}_b] & = & i\theta_{ab} \quad , \\ [\hat{p}_a, \hat{p}_b] & = & 0 \quad , \qquad\qquad a, b, \ldots \in \{1, 2\} \quad , \\ [\hat{x}_a, \hat{p}_b] & = & i\hbar\delta_{ab} \quad , \end{array} \tag{1}$$

where θ_{ab} is real and antisymetric (i.e. $\theta_{ab} = \theta\epsilon_{ab}$). The spatial non-commutative parameter θ is of dimension of $(length)^2$, so $\sqrt{\theta}$ may be considered as the fundamental length (Planck length?). If θ goes to zero we obtain the standart Heisenberg algebra commutation relations.

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Suitable realization of commutation relations between the position operators (first line in (1)) is given by the $\star$-product (Moyal product) [4] defined as follows

$$(f \star g)(x) = \exp\left[\frac{i}{2}\theta_{ab}\partial_{x_a}\partial_{y_b}\right] f(x)g(y)|_{x=y} \quad . \tag{2}$$

The wave functions we take as $\psi(\vec{x})$ and the operators $\hat{\vec{x}}$ and $\hat{\vec{p}}$ are realized as follows

$$\hat{x}_a\psi(\vec{x}) = x_a \star \psi(\vec{x}) \quad , \qquad \hat{p}_a\psi(\vec{x}) = -i\hbar\partial_a\psi(\vec{x}) \quad .$$

The quantization procedure is the same as in the standard quantum mechanics. We replace the classical observables on the phase space $A(\vec{p},\vec{x})$ by the self-adjoint operators $\hat{A} = A(\hat{\vec{p}},\hat{\vec{x}})$ which act on suitable Hilbert space. The ordering problem is like that one in the standard quantum theory. The Hilbert space can consistently be taken to be exactly the same as the Hilbert space of the corresponding commuting system, for example squared integrables functions on the plane with the standart Lebesgue measure $L^2(R^2)$. The time evolution is given by the Schrödinger equation

$$i\hbar\partial_t|\psi\rangle = \hat{H}|\psi\rangle \quad ,$$

where $\hat{H} = H(\hat{\vec{p}},\hat{\vec{x}})$ is the Hamiltonian. The only nontrivial part of such a formulation is to give the Hamiltonian. In what follows we will consider two dimensional hydrogen-like atom and isotropic linear harmonic oscillator.

2 Hydrogen-like atom

Two dimensional hydrogen-like atom in NCQM is defined by the following Hamiltonian (to simplify calculations we will use the system of units in which the mass of electron and Planck constant are equal to a unit and we are using Einsten's summation convention in latin indeces)

$$\hat{H} = \frac{1}{2}\hat{p}_a\hat{p}_a + U_0 \ln\left(\frac{\sqrt{\hat{x}_a\hat{x}_a}}{r_0}\right) \equiv \frac{1}{2}\hat{p}_a\hat{p}_a + U(\hat{\vec{x}}) \quad , \tag{3}$$

where U_0 and r_0 are the positive constants. We define the new operators $\tilde{x}_a$, $\tilde{p}_a$

$$\begin{array}{rcl} \tilde{x}_a & = & \hat{x}_a + \frac{1}{2}\theta_{ab}\hat{p}_b \quad , \\ \tilde{p}_a & = & \hat{p}_a \quad , \end{array} \tag{4}$$

which satisfy the usual canonical commutation relations

$$\begin{array}{rcl} [\tilde{x}_a,\tilde{x}_b] & = & 0 \quad , \\ [\tilde{p}_a,\tilde{p}_b] & = & 0 \quad , \\ [\tilde{x}_a,\tilde{p}_b] & = & i\delta_{ab} \quad . \end{array} \tag{5}$$

The representation of the above commutation relations in the $L^2(R^2)$ is well known:

$$(\tilde{x}_a f)(\vec{x}) = x_a f(\vec{x}) \quad , \qquad (\tilde{p}_a f)(\vec{x}) = -i\partial_a f(\vec{x}) \quad . \tag{6}$$

If we replace hat operators by tilde operators in the Hamiltonian (3) and expand the potential to the Taylor series, we obtain the first θ - order time independent Schrödinger equation in polar coordinates (r,ϕ)

$$\Delta\psi = 2\left[U_0 \ln\left(\frac{r}{r_0}\right) - \epsilon + \theta\frac{iU_0}{2r^2}\partial_\phi\right]\psi \quad , \tag{7}$$

where ϵ is the Hamiltonian eigenvalue. We note that $-i\partial_\phi$ is the z component (L_z) of the angular momentum operator. Standard variables separation procedure $\psi(r,\phi) = R(r)\exp(im\phi)$ leads to the radial Schrödinger equation

$$\frac{1}{r}(rR')' + \left[2(\epsilon - U(r)) - \frac{m^2 - mU_0\theta}{r^2}\right]R = 0 \quad , \tag{8}$$

with m (orbital quantum number) is an arbitrary integer.
In the case of $\theta = 0$ we obtain radial equation for 2D Hydrogen-like atom in commutative QM. It is an important fact that the structure of equation (8) will not change. So we can state that if

$$\epsilon^C = \epsilon^C(n, m^2) \quad , \tag{9}$$

is the energetical spectrum in the commutative case then in NCQM the spectrum is given by

$$\epsilon(n, m^2) = \epsilon^C(n, m^2 - mU_0\theta) \quad , \tag{10}$$

where n is the principal quantum number. We have derived the approximative formulae for the commutative spectrum (see Appendix). The results are
(i.) For the states with $|m| \gg n$, $n = 1, 2, \ldots$

$$\epsilon(n, m^2) = \frac{U_0}{2}\left\{1 + \ln\left[\frac{m^2 - mU_0\theta - 1/4}{r_0^2 U_0}\right] + \sqrt{2}\frac{n - 1/2}{\sqrt{m^2 - mU_0\theta - 1/4}}\right\} \quad . \tag{11}$$

(ii.) For high excited ($n \gg 1$) states with zero orbital momentum ($m = 0$)

$$\epsilon(n, m^2 = 0) = \frac{U_0}{2}\left\{\ln\left[\frac{\pi}{2U_0 r_0^2}\right] + 2\ln(2n + 1/2)\right\} \quad . \tag{12}$$

We note that the similar method is appropriate for any radial symmetrical potential.

3 Linear harmonic oscillator

Let us consider now a slight generalization of the commutation relations (1)

$$\begin{aligned} [\hat{x}_a, \hat{x}_b] &= i\theta_{ab} \quad , \\ [\hat{p}_a, \hat{p}_b] &= i\kappa_{ab} \quad , \\ [\hat{x}_a, \hat{p}_b] &= i\hbar\delta_{ab} \quad . \end{aligned} \tag{13}$$

The role of $\kappa_{ab} = \kappa\epsilon_{ab}$ is similar as θ_{ab} with the difference between them that κ is of dimension of $(momentum)^2$ and $\sqrt{\kappa}$ plays the role of fundamental momentum.
We would like to find a general linear transormation $(\hat{\vec{x}}, \hat{\vec{p}}) \rightarrow (\tilde{\vec{x}}, \tilde{\vec{p}})$ so that tilde operators satisfy canonical commutation relations. We have shown that there exist an infinite number of such transformations labeled by two types of indices $\mu \in R \setminus \{0\}$ and $\sigma \in \{+1, -1\}$. The transformation in question has a form

$$\begin{aligned} \tilde{x}_a^{\mu\sigma} &= \alpha_\sigma^\mu \hat{x}_a + \beta^\mu \epsilon_{ab} \hat{p}_b \quad , \\ \tilde{p}_a^{\mu\sigma} &= \mu \hat{p}_a - \xi_\sigma^\mu \epsilon_{ab} \hat{x}_b \quad , \end{aligned} \tag{14}$$

where the coefficients are

$$\alpha_\sigma^\mu = \left(1 + \sigma\sqrt{1 - \frac{\kappa\theta}{\hbar^2}}\right) \frac{1}{2\mu\left(1 - \frac{\kappa\theta}{\hbar^2}\right)} \quad , \tag{15}$$

$$\xi_\sigma^\mu = \frac{\hbar\mu}{\theta}\left(1 - \sigma\sqrt{1 - \frac{\kappa\theta}{\hbar^2}}\right) \quad , \tag{16}$$

$$\beta^\mu = \frac{\theta}{2\mu\hbar\left(1 - \frac{\kappa\theta}{\hbar^2}\right)} \quad . \tag{17}$$

The transformations (4) can be acquired from these ones putting $\hbar = 1$, $\mu = 1$, $\sigma = 1$ and $\kappa = 0$.
The operator realization of tilde operators in $L^2(R^2)$ is the standard one given by (6). The inverse transformations are

$$\begin{aligned} \hat{x}_a^{\mu\sigma} &= A_\sigma^\mu \tilde{x}_a - B_\sigma^\mu \epsilon_{ab} \tilde{p}_b \quad , \\ \hat{p}_a^{\mu\sigma} &= C_\sigma^\mu \tilde{p}_a + D_\sigma^\mu \epsilon_{ab} \tilde{x}_b \quad , \end{aligned} \tag{18}$$

where the constants are

$$A_\sigma^\mu = \mu\sigma\sqrt{1 - \frac{\kappa\theta}{\hbar^2}} \quad , \tag{19}$$

$$B_\sigma^\mu = \frac{\theta\sigma}{2\hbar\mu\sqrt{1 - \frac{\kappa\theta}{\hbar^2}}} \quad , \tag{20}$$

$$C_\sigma^\mu = \frac{\sigma + \sqrt{1 - \frac{\kappa\theta}{\hbar^2}}}{2\mu\sqrt{1 - \frac{\kappa\theta}{\hbar^2}}} \quad , \tag{21}$$

$$D_\sigma^\mu = \frac{\hbar\mu}{\theta}\sqrt{1 - \frac{\kappa\theta}{\hbar^2}}\left(\sigma - \sqrt{1 - \frac{\kappa\theta}{\hbar^2}}\right) \quad . \tag{22}$$

We stress that eqs. (15)-(17) and (19)-(22) request $\kappa\theta < \hbar^2$, otherwise the self-adjointness of the position and momentum operators will be violated.

Let us consider the isotropic linear harmonic oscillator Hamiltonian in the NCQM (to simplify the calculations we put $\hbar = m = 1$)

$$\hat{H} = \frac{1}{2}\hat{p}_a\hat{p}_a + \frac{1}{2}\omega^2\hat{x}_a\hat{x}_a \quad . \tag{23}$$

We express the hat operators in the LHO Hamiltonian in terms of the tilde operators using the formulae (18)

$$\begin{aligned}\hat{H} \;=\; & \frac{1}{2}\left[(C^{\mu}_{\sigma})^2 + \omega^2(B^{\mu}_{\sigma})^2\right]\tilde{p}_a\tilde{p}_a + \frac{1}{2}\left[(D^{\mu}_{\sigma})^2 + \omega^2(A^{\mu}_{\sigma})^2\right]\tilde{x}_a\tilde{x}_a + \\ & \left[C^{\mu}_{\sigma}D^{\mu}_{\sigma} + \omega^2 A^{\mu}_{\sigma}B^{\mu}_{\sigma}\right]\tilde{p}_a\epsilon_{ab}\tilde{x}_b \quad .\end{aligned} \tag{24}$$

The last term in the Hamiltonian is proportional to the z component of the angular momentum ($L_z = \epsilon_{ab}\tilde{x}_a\tilde{p}_p$). The spectrum $\epsilon_{n_1n_2}$ of this type of Hamiltonian is well known, so we will only write down the result

$$\epsilon_{n_1n_2} = \sqrt{\omega^2 + \frac{1}{4}\left(\kappa - \omega^2\theta\right)^2}(n_1 + n_2 + 1) - \frac{1}{2}(\kappa + \theta\omega^2)(n_1 - n_2) \quad , \tag{25}$$

where $n_1, n_2 = 0, 1, 2\ldots$.

The LHO Hamiltonian (24) depends (via A, B, C, D which are given by (19)-(22)) on the parameters μ and σ but the spectrum (25) is, of course, independent on this parameters. In the case of κ and θ tend to zero we recover the standard QM spectrum for the 2D LHO.

4 Conclusion

In this paper we have presented the results on the two 2D systems within non-commutative quantum mechanics for the hydrogen-like atom and isotropic LHO. We have obtained the corrections to the classical (commutative) spectra of related Hamiltonians. We note that 2D hydrogen-like atom has the analogy in 3D - motion of a charged particle around the homogenous charged straight line.

The odds are that the main goal of the NCQM is to find a measurable effect. Unfortunately we can not hope that our corrections to the energy levels of the systems in question may be experimentally verified because of its dimensionality (LHO) and experimental complications with the realization of 3D analogy of hydrogen-like atom. In [6] the modification of energy levels and Lamb shift for 3D hydrogen-like atom due to the presence of the non-commutative plane in R^3 was presented as "measurable". But, this type of space non-commutativity is not applicable if we insist on the isotropy of space. In this spite it would be desirable to use 3D space non-commutativity which preserves rotational symmetry of the hydrogen-like atom problems.

At the beginning of our treatment to the NCQM are the nontrivial commutation relations between the position operators (see in (1)). We note the the noncommutativity of the (only "possible") positions operators is well known from the relativistic quantum mechanics [7]. We have shown the new academic analogy with relativistic quantum mechanics in the (perturbative) formula (11). For very large values of $U_0\theta$ one can obtain for properly small m nonzero imaginary part of energy. This effect is known from Klein-Gordon (and Dirac too) theory of hydrogen-like atom.

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Appendix

In this section we briefly describe how to find the formulae (11) and (12). Let us start with the equation (8), which structure does not change putting $\theta = 0$ (the commutative case).

Let us first analyze the case of $m \neq 0$. We introduce to eq. (8) a new function $\chi(r) = \sqrt{r}R(r)$ then we get

$$\chi'' + [2\epsilon - 2U_{eff}]\,\chi = 0 \quad , \qquad \chi \in L^2(R^+) \quad , \tag{26}$$

where the effective potential U_{eff} is given as

$$U_{eff} = U + \frac{m^2 - 1/4}{2r^2} \quad .$$

We expand U_{eff} into the powers of $(r - r_k)$, where $r_k = [(m^2 - 1/4)/U_0]^{1/2}$ is the point at which U_{eff} is minimal. Harmonic approximation expansion around minima gives

$$U^h_{eff}(r) = U_{eff}(r_k) + \frac{1}{2}\Omega^2(r - r_k)^2 \quad , \qquad \Omega^2 = 2\frac{U_0^2}{m^2 - 1/4} \quad . \tag{27}$$

It may be easily shown that the harmonic approximation is valid only if the following inequality is fulfilled

$$\left| \frac{U_{eff} - U^h_{eff}}{U^h_{eff}} \right| \approx \frac{5}{3}\frac{(2n+1)^{1/2}}{(2m^2 - 1/2)^{1/4}} \ll 1 \quad , \tag{28}$$

where $n = 1, 2, \ldots$ is the principal quantum number. For the LHO with the effective potential (27) one obtains eq.(11) with $\theta = 0$.

In the second step we analyze the high excited states with zero angular momentum m. Substitution

$$v = \ln\left(\frac{r}{r_0}\right) - \frac{\epsilon}{U_0} \quad , \qquad v \in R \quad ,$$

in (8) leads to

$$\hat{R}''(v) - \left[\Xi v e^{2v}\right]\hat{R}(v) = 0 \quad , \quad \Xi(\epsilon) = 2U_0 r_0^2 \exp\left(\frac{2\epsilon}{U_0}\right) \quad . \tag{29}$$

Now we will investigate the asymtotic solutions of the above boundary problem. In the case of $v \to -\infty$ the only acceptable solution is

$$\hat{R}(v)_- = const \quad . \tag{30}$$

In the other extreme case of $v \to +\infty$ we have the WKB [5] solution

$$\hat{R}(v)_+ \sim V_0^{-1/4} \exp\left(-V_0^{1/2}\right) \quad , \qquad V_0 = \Xi v e^{2v} \quad . \tag{31}$$

It is important that the WKB limit is applicable not only for $v \to +\infty$, but for $v > v_m$, as well, where v_m is determined by the WKB condition

$$\frac{1}{2V_0} \mid \frac{d(V_0)^{1/2}}{dv} \mid \ll 1 \quad . \tag{32}$$

For high excited states with $\Xi \gg 1$ v_m can be approximated by

$$|v_m| \sim \ln(\Xi) \qquad \text{and} \qquad v_m < 0 \quad . \tag{33}$$

The full WKB function $\hat{R}^{WKB}$ is then of the form

$$\hat{R}(v)^{WKB} = \begin{cases} C_0 p^{-1/2} \sin\left(\int_v^0 p' dv' + \pi/4\right) & ; \quad v_m < v < 0 \quad , \\ \\ \frac{1}{2} C_0 q^{-1/2} \exp\left(-\int_0^v q' dv'\right) & ; \qquad v > 0 \quad , \end{cases} \tag{34}$$

where $p = (-V_0)^{1/2}$ and $q = (V_0)^{1/2}$. The continuity condition of the wave function in v_m leads to the following analogy of the Bohr-Sommerfeld quantization rule

$$\sqrt{\Xi} \int_{-\infty}^{0} |v|^{1/2} e^v dv = \pi(n + 1/4) \quad , \tag{35}$$

from which we, using (31), obtian instantly the spectrum formula (12).

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